Let f be a 2π-periodic function.

Question: Under which assumption, the Fourier Series of f converges absolutely?

I. $f \in C^1 \Longrightarrow$ Absolute Convergence of Fourier Series

Proof:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{in}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx \qquad \text{integral by part}$$

$$= \frac{1}{in}\,\hat{f'}(n)$$

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| = \sum_{n\in\mathbb{Z}} \left|\frac{1}{in}\hat{f'}(n)\right|$$

$$\le \left(\sum_{n\in\mathbb{Z}} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} |\hat{f'}(n)|^2\right)^{\frac{1}{2}}$$

$$\le \left(\sum_{n\in\mathbb{Z}} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx\right)^{\frac{1}{2}} \qquad \text{by Parseval's Identity}$$

$$< \infty$$

$\square$

II. f Lipschitz $\Rightarrow$ Absolute Convergence of its Fourier Series

Recall the definition of Lipschitz: $|f(x)-f(y)| \le C|x-y|$, i.e., 1-Hölder. By Tutorial 3, $|\hat{f}(n)| = O(\frac{1}{|n|})$.

However, $\sum \frac{1}{n}$ diverges.

Proof:

Step 1: For any $h>0$, $\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 = O(h^2)$

Pf: Let $g(x) = f(x+h) - f(x-h)$

Then
$$\hat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x) e^{-inx} dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h) e^{-inx} dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-h) e^{-inx} dx$$
$$= e^{inh}\hat{f}(n) - e^{-inh}\hat{f}(n)$$
$$= 2i \sin nh \, \hat{f}(n)$$

Then
$$4\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 = \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2 dx \quad \text{Parseval's Identity}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^2 dx$$

$$\lesssim h^2 \qquad \text{Lipschitz}$$

Step 2: $\displaystyle\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 = O\left(\frac{1}{4^p}\right)$

Pf: Let $h = \frac{\pi}{2^{p+1}}$.

For n with $2^{p-1} < |n| \le 2^p$,

we have $\frac{\pi}{4} \le |nh| \le \frac{\pi}{2}$, thus

$$\frac{1}{2} \le |\sin nh|^2 \le 1$$

Then $\displaystyle\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 \le 2\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 |\sin nh|^2$

$$\lesssim 2\left(\frac{\pi}{2^{p+1}}\right)^2 \qquad \text{Step 1}$$

$$\lesssim \frac{1}{4^p}$$

Step 3: $\displaystyle\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)| \lesssim \left(\frac{1}{\sqrt{2}}\right)^p$

$$\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)| \le \left(\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2\right)^{\frac{1}{2}} \left(\sum_{2^{p-1}<|n|\le 2^p} 1\right)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{2^p}\,(2^p)^{\frac{1}{2}}$$

$$\lesssim \left(\frac{1}{\sqrt{2}}\right)^p$$

Step 4: $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$

Pf: $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{p=1}^{\infty} \sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|$

$$\lesssim \sum_{p=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^p$$

$$< \infty$$

$\square$

III. f α-Hölder $(\alpha > \frac{1}{2}) \Longrightarrow$ Absolute Convergence of its Fourier Series.

Proof: Step 1: $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 |\sin hn|^2 \lesssim h^{2\alpha}$

Step 2: $\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 = O\left(\frac{1}{4^{\alpha p}}\right)$

Step 3: $\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)| = O\left(\frac{1}{2^{(\alpha-\frac{1}{2})p}}\right)$

Step 4: $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \le \sum_{p=1}^{\infty} \left(\frac{1}{2^{\alpha-\frac{1}{2}}}\right)^p < \infty$.

$\square$